INRNE-TH-97/6

On the Connection between Pauli—Villars and Higher Derivative Regularizations

M.N. Stoilov* ⋆

Institute for Nuclear Research and Nuclear Energy
Boul. Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria

Abstract. We show that in some cases the gauge invariant Pauli—Villars and higher (covariant) derivatives regularizations are equivalent.

10 June 97

* e-mail: mstoilov@inrne.acad.bg

⋆ Supported in part by Bulgarian National Scientific Foundation under contract Φ-401-94

Higher covariant derivatives and gauge invariant Pauli—Villars regularizations have a quite special place in the long list of regularizations used in Quantum Field Theory. First, a combination of these regularizations is used to prove the renormalizability of Yang—Mills theories [1]. Second, they are the only ones which could be incorporated into the Lagrangian of the model as additional local terms. In this paper our aim is to show that these regularizations have something more in common — in fact, in some cases they are just two different forms of a same regularization.

It seems that higher derivatives (HD) regularization originates from the usual Pauli—Villars (PV) one. The latter prescribes to replace in loop calculations the propagator $(\not{p}-m)^{-1}$ with $(\not{p}+m)\left((p^2-m^2)^{-1}+\sum_{j=1}^{k} c_j(p^2-m_j^2)^{-1}\right)$. (Here we deal with spinors only. However, the same approach could be applied to field with any spin and statistics.) The constants c_j and m_j are such that

$$1+\sum_{j=1}^{k} c_j=0, \quad m^2+\sum_{j=1}^{k} c_j m_j^2=0, \quad \text{etc.} \ . \tag{1}$$

Equations (1) allow c_j to be integer. In this case the regularized propagator could be put in the form $\frac{1}{g}(\not{p}-m)^{-1}\prod_{j=1}^{k}(p^2-m_j^2)^{-1}$, where $(-1)^k g^{-1}=\prod_{i=1}^{k} m_i^2+\sum_j^k c_j m^2\prod_{i\neq j} m_i^2$. On the other hand this propagator could be viewed as obtained from a Lagrangian with the following free term $L^{free}=g\int \mathrm{d}^3x\, \bar{\psi}(\,i\,\not{\partial}\,-m)\prod_{j=1}^{k}(-\partial^2-m_j^2)\psi$. Replacing this particular form of L^{free} with the most general (polynomial) expression and usual derivatives with covariant ones one obtains a (variant of) higher covariant derivatives regularization for spinor field. The spinor part of the Lagrangian in this case is (here g is a constant with dimension $mass^{-k}$ and A is the gauge potential)

$$L=g\int \mathrm{d}^3x\, \bar{\psi}(\,i\,\not{\partial}+\not{A}-m)\prod_{j=1}^{k}(\,i\,\not{\partial}+\not{A}-m_j)\psi. \tag{2}$$

One of the possible viewpoints to the gauge invariant PV regularization is that in divergent diagrams one regularises a whole spinor loop at a time adding and subtracting the same diagram (with some integer coefficients c_i) but with different masses in the propagators forming the loop. It is possible to write down a Lagrangian which reproduce automatically this scheme and it has the form

$$L=\int \bar{\psi}(\,i\,\not{\partial}+\not{A}-m)\psi+\sum_i\sum_{j=1}^{c_i}\bar{\psi}_{ij}(\,i\,\not{\partial}+\not{A}-m_i)\psi_{ij}+\sum_k\sum_{j=1}^{|c_k|}\bar{\phi}_{kj}(\,i\,\not{\partial}+\not{A}-m_k)\phi_{kj}. \tag{3}$$

The sum in the second term in (3) is over positive coefficients c and so, the extra fields ψ_{ij} are with Fermi statistics; the sum in the third term is over negative coefficients and thus ϕ_{kj} are Bose fields. Usually, extra fields have one and the same mass M. This has some advantages, however, using different masses one can fix $|c_i| = 1$. In this case (3) takes the form

$$L = \int \mathrm{d}^3x \; \bar{\psi}(\, i \,\partial\!\!\!/ + A\!\!\!/ - m)\psi + \sum_i \bar{\psi}_i(\, i \,\partial\!\!\!/ + A\!\!\!/ - m_i)\psi_i + \sum_k \bar{\phi}_k(\, i \,\partial\!\!\!/ + A\!\!\!/ - m_k)\phi_k. \quad (4)$$

Our goal in this letter is to show that model with Lagrangian (2) is equivalent (in some cases) to that with Lagrangian (4). The first step is to transform (2) into first order Lagrangian. After that we represent the HD fermionic ghosts [2] arising in step one as boson ghosts. A short note is needed at this stage before go further. The extra fields in the gauge invariant PV regularized Lagrangians (3) and (4) are unphysical and *a priori* there are not sources for them in the generating functional of the model [1]. Something similar has to take place in the models with HD — it is easy to show that provided $m_i \neq m_j \quad \forall \quad i \neq j$ the general solution of a HD equation $\prod_j (\, i \,\partial\!\!\!/ \, - m_j)\psi = 0$ is $\psi = \sum_j \psi_j$ where $(\, i \,\partial\!\!\!/ \, - m_j)\psi_j = 0$. Therefore, after quantization ψ describes a set of ordinary fields. However, only one of these fields is physical and so, there has to be source only for it. (This is obvious if one looks at HD regularization as a variant of the usual PV one.) A possible way to achieve this is to enforce 'by hands' $(\, i \,\partial\!\!\!/ \, - m)|\mathrm{ph} >= 0$; another possibility is discussed in [3]. In fact it does not matter for us how the problem is cured. The only important thing is that there are not sources for extra fields in HD case too. As a consequence, we can work simply with the Lagrangians (2) and (4) and not with corresponding generating functionals.

We begin our considerations on the conversion of HD to first order Lagrangian with a simple example of second order (in derivatives) free Lagrangian for spinor field

$$\begin{aligned} L'' &= g \int \mathrm{d}^3x \; \bar{\psi}(\, i \,\partial\!\!\!/ \, - m_1)(\, i \,\partial\!\!\!/ \, - m_2)\psi \\ &= g \int \mathrm{d}^3x \; \bar{\psi}(-i \overleftarrow{\partial\!\!\!/}) \, (\, i \overrightarrow{\partial\!\!\!/} \,)\psi - \frac{g}{2}(m_1 + m_2)\bar{\psi} \, \overleftrightarrow{\partial\!\!\!/} \, \psi + g m_1 m_2 \bar{\psi}\psi. \end{aligned} \quad (5)$$

As it was mentioned above if $m_1 \neq m_2$ (for definitness we use $m_1 > m_2$) the solution of the equation of motion is $\psi = \psi_1 + \psi_2$, where $(\, i \,\partial\!\!\!/ \, - m_i)\psi_i = 0$. Moreover, any dynamical invariant (energy-momentum, charge, etc.) is a sum of the corresponding invariants for usual spinor fields with mass m_1 and m_2 (one of them with minus sign). These facts

suggest that L itself also could be presented as a sum of usual fermionic Lagrangians. We shall demonstrate this using suitable Legendre transformation. The procedure is an analogue of the one used for Lagrangian derivation of Hamiltonian equations and is often used in the analysis of HD theories [4]. Let us introduce the quantities

$$\begin{aligned} a &\equiv \; i\overrightarrow{\not\partial}\,\psi \\ \bar{a} &\equiv \bar{\psi}(-i\overleftarrow{\not\partial})\;. \end{aligned}$$

The functional variations of L with respect to a and $\bar{a}$ are

$$\begin{aligned} \pi &\equiv \frac{\delta L''}{\delta \bar{a}} = g\bar{\psi}(-i\overleftarrow{\not\partial})\; - \frac{g}{2}(m_1+m_2)\bar{\psi} \\ \bar{\pi} &\equiv \frac{\delta L''}{\delta a} = g\; i\overrightarrow{\not\partial}\,\psi - \frac{g}{2}(m_1+m_2)\psi. \end{aligned}$$

These identities are used to express a as a function of π and ψ. The Legendre transform of L'' with respect to a and $\bar{a}$ is

$$\begin{aligned} \Lambda\left[\bar{\pi},\bar{\psi},\pi,\psi\right] &= \int \mathrm{d}^3x\;(\bar{\pi}a+\bar{a}\pi) - L'' \\ &= \int \mathrm{d}^3x\; g(\frac{1}{g}\bar{\pi}+\frac{m_1+m_2}{2}\bar{\psi})(\frac{1}{g}\pi+\frac{m_1+m_2}{2}\psi) - gm_1m_2\bar{\psi}\psi \end{aligned}$$

and the first order Lagrangian governing the dynamics of our model is

$$\begin{aligned} L' &= \int \mathrm{d}^3x\;\bar{\pi}\;i\overrightarrow{\not\partial}\,\psi + \bar{\psi}(-i\overleftarrow{\not\partial})\;\pi - \Lambda \\ &= \int \mathrm{d}^3x\;\bar{\pi}\;i\overrightarrow{\not\partial}\,\psi + \bar{\psi}(-i\overleftarrow{\not\partial})\;\pi - \frac{1}{g}\bar{\pi}\pi - \frac{m_1+m_2}{2}(\bar{\pi}\psi+\bar{\psi}\pi) - \frac{g}{4}(m_1-m_2)^2\bar{\psi}\psi. \end{aligned} \tag{6}$$

It is easy to check that the equations of motion for ψ and $\bar{\psi}$, following from (6) coincide with those from (5). Now we would like to diagonalize L'. For this purpose we introduce the linear combination

$$\begin{pmatrix}\phi\\ \chi\end{pmatrix} = U\begin{pmatrix}\pi\\ \psi\end{pmatrix}$$

where U is some 2×2 complex matrix such that $|\mathrm{det}U|^2 = 1$. Fixing the elements u_{ij} of the U so that

$$\begin{aligned} u_{11} &= \pm\frac{1}{2}g(m_1-m_2)u_{21} \\ u_{12} &= \mp\frac{1}{2}g(m_1-m_2)u_{22} \\ |u_{21}|^2 &= 1/|g(m_1-m_2)| = |u_{22}|^2 \end{aligned}$$

L' takes the form

$$L' = \frac{g}{|g|}\int \mathrm{d}^3x\ \frac{i}{2}\bar{\phi}\ \overleftrightarrow{\not{\partial}}\,\phi - m_1\bar{\phi}\phi - \frac{i}{2}\bar{\chi}\ \overleftrightarrow{\not{\partial}}\,\chi + m_2\bar{\chi}\chi. \tag{7}$$

We see that the our initial second order theory can be described by a difference of two usual spinor Lagrangians for two (not interacting) Fermi fields. The field χ which Lagrangian enters (7) with minus sign is a ghost field [2]. It is possible to change this bad sign in the kinetic term but as we shall see this is of little use. The change can be achieved by a suitable antiunitary transformation (of the time-reverse type) applied on χ field. Namely, let us denote with $'$ the quantities after transformation (as usual, the antiunitary transformed of any operator A is $A' = (U^{-1}AU)^{\dagger}$). For χ field we have:

$$\begin{aligned}\chi' &= \eta\bar{\chi}T\\ \bar{\chi}' &= \eta^{*}T^{-1}\chi, \quad |\eta|^2 = 1.\end{aligned}$$

Here T is some matrix, we want to satisfy $T^{\dagger} = T^{-1}$ and $\gamma^0 T\gamma^0 = T$. The kinetic term for χ field changes the sign provided that in additional

$$T\overset{\mathrm{T}}{\gamma}{}^{\mu}T^{-1} = \gamma^{\mu}. \tag{8}$$

The matrices $\overset{\mathrm{T}}{\gamma}{}^{\mu}$ satisfy the identities for γ-matices, so there is a (unitary) matrix T with desired property (8). Thus, formally, we can write L' as a sum of two ordinary spinor Lagrangians but with different signs of the mass terms. However, the transformation used is not unitary and therefore, on quantum level the two theories will differ. If we want to keep the contact with higher derivative theory, we should quantize one of the fields in a non standard way thus coming back to the form (7) of L'.

Now we proceed the $2n$-th order case in the presence of gauge interaction. It is always possible to write its spinor Lagrangian in the form

$$L^{(2n)} = g\int \mathrm{d}^3x\ \bar{\psi}\overleftarrow{\mathcal{D}}\,(\overrightarrow{\mathcal{D}} + \overrightarrow{\mathcal{A}}\,)\psi \tag{9}$$

where

$$\begin{aligned}\overrightarrow{\mathcal{D}} &= \prod_{j=1}^{n}(\,i\,\overrightarrow{\not{\partial}} + \not{A} - m_j),\\ \overleftarrow{\mathcal{D}} &= \prod_{j=1}^{n}((-i\,\overleftarrow{\not{\partial}}) + \not{A} - m_j),\end{aligned}$$

and $\mathcal{A}$ is some operator of order $k < n$ which commutes with $\mathcal{D}$. As a consequence every function of $\mathcal{A}$ and $\mathcal{A}^{-1}$ (which should be understand as a power series of $\mathcal{A}$) commutes also with $\mathcal{D}$. The variables we use in Legendre transformation are

$$a \equiv \overrightarrow{\mathcal{D}}\,\psi, \quad \bar{a} \equiv \bar{\psi}\overleftarrow{\mathcal{D}}\,;$$

the corresponding momenta are

$$\bar{\pi} = g(\bar{a} + \frac{1}{2}\bar{\psi}\overleftarrow{\mathcal{A}}\,), \quad \pi = g(a + \frac{1}{2}\overrightarrow{\mathcal{A}}\,\psi),$$

and the n-th order Lagrangian, equivalent to $L^{(2n)}$ is

$$L^{(n)} = \int \mathrm{d}^3x\ \bar{\pi}\overrightarrow{\mathcal{D}}\,\psi + \bar{\psi}\overleftarrow{\mathcal{D}}\,\pi - g(\frac{1}{g}\bar{\pi} - \frac{1}{2}\bar{\psi}\overleftarrow{\mathcal{A}}\,)(\frac{1}{g}\pi - \frac{1}{2}\overrightarrow{\mathcal{A}}\,\psi) \tag{10}$$

Again, as in the second order case, the equations of motion for ψ and $\bar{\psi}$ following from (10) reproduce the ones from (9).

Up to this point we simply repeat the procedure used in the second order case. The not so straightforward step is the diagonalization of the $L^{(n)}$. Consider the following variables' change

$$\begin{aligned} \chi &= \sqrt{\frac{g}{\overrightarrow{\mathcal{A}}}}(\frac{1}{g}\pi + \frac{1}{2}\overrightarrow{\mathcal{A}}\,\psi), \\ \phi &= \sqrt{\frac{g}{\overrightarrow{\mathcal{A}}}}(\frac{1}{g}\pi - \frac{1}{2}\overrightarrow{\mathcal{A}}\,\psi), \end{aligned} \tag{11}$$

and analogously for $\bar{\chi}$ and $\bar{\phi}$. The Berezian of this change of variables is 1 and so, (11) leaves the integration measure in the generating functional invariant. In the new variables $L^{(n)}$ takes the form

$$L^{(n)} = \bar{\chi}\overrightarrow{\mathcal{D}}\,\chi - \bar{\phi}(\overrightarrow{\mathcal{D}}\, + \overrightarrow{\mathcal{A}}\,)\phi. \tag{12}$$

Note, that deriving (12) we have not used the fact that k (the order of $\mathcal{A}$) is less than n (the order of $\mathcal{D}$). The only thing that is really important is that $k \neq n$. Therefore, we could consider the case $k = n + 1$ and thus to cover all HD Lagrangians.

Repeating the above steps sufficiently times we could present the HD Lagrangian (2) of arbitrary order n as a sum of n first order Lagrangians (with altering signs) for n independent Fermi fields

$$L = \int \mathrm{d}^3x\ \sum_i \bar{\psi}_i(\,i\,\partial\!\!\!/ + A\!\!\!/ - m_i)\psi_i - \sum_i \bar{\chi}_i(\,i\,\partial\!\!\!/ + A\!\!\!/ - m'_i)\chi_i. \tag{13}$$

Same is the structure of the gauge invariant PV regularized Lagrangian. The only difference is that in (4) there are not spinor terms with opposite signs, but Bose terms (with correct sign). Now we want to show that this difference could be removed. For this we use the so called 'collective field method' [5] — we introduce extra gauge freedom in the model and then fix it. In fact, we fix the gauge in two different ways. The first gives the linear decomposition (13) of the HD Lagrangian, the second - the gauge invariant PV regularization. To clarify the idea let us consider a simple example with only one (spinor) field and Lagrangian

$$L = \int \mathrm{d}^3 x\, \bar{\psi} \mathcal{D} \psi. \tag{14}$$

Here $\mathcal{D}$ is some operator we shall not specify and we suppose there are no sources for ψ and $\bar{\psi}$ in the generating functional of the theory. Let us introduce extra (collective) field ϕ, so that the Lagrangian becomes

$$L = (\bar{\psi} + \bar{\phi}) \mathcal{D} (\psi + \phi). \tag{15}$$

This Lagrangian possesses an extra local symmetry

$$\begin{aligned} \delta\psi &= -\rho \\ \delta\phi &= \rho, \end{aligned} \tag{16}$$

where ρ is an arbitrary spinor function. Following [6] we introduce an auxiliary field λ and a ghost pair $\{c, \bar{c}\}$ for this gauge symmetry (the ghosts are bosons due to the spinor character of the ρ). After gauge fixing the Lagrangian is invariant under rigid BRST symmetry. The infinitesimal BRST transformation of the fields we are interested in are (here ϵ is the parameter of the transformation)

$$\begin{aligned} \delta_Q \psi &= -c\epsilon, \\ \delta_Q \phi &= c\epsilon, \\ \delta_Q \bar{c} &= \lambda, \\ \delta_Q \lambda &= 0. \end{aligned}$$

The BRST invariant Lagrangian has the form [6]

$$L_{BRST} = L + \delta_Q (\bar{c} \varphi),$$

where φ is the gauge fixing condition and L is that of formula (15). Choosing $\varphi = \phi$ we get

$$\delta_Q(\bar{c}\varphi) = \lambda\phi + \bar{c}c$$

Thus the ghosts trivially decouple from the dynamics of the system, the field ϕ is set to zero and we restore the initial model with Lagrangian (14). If we choose $\varphi = \mathcal{D}\phi$ the result reads

$$L_{BRST} = \bar{\psi}\mathcal{D}\psi + \bar{c}\mathcal{D}c. \tag{17}$$

The same result is obtained if we consider instead of gauge transformation (16) the following one

$$\delta\psi = -\mathcal{D}\rho$$

$$\delta\phi = \mathcal{D}\rho$$

with gauge condition $\varphi = \phi$. In our next step we introduce the collective field in a slightly different way. We replace (14) with

$$L = \int \mathrm{d}^3x \; (\bar{\psi} + \bar{\phi}\mathcal{A})\mathcal{D}(\psi + \mathcal{A}\phi), \tag{18}$$

where $\mathcal{A} = \sqrt{\mathcal{D}^{-1}}$. ($\mathcal{A}$ should be understand again as a series of $\mathcal{D}$). The extra symmetry in this case is

$$\delta\psi = -\sqrt{\mathcal{D}}\rho$$

$$\delta\phi = \mathcal{D}\rho,$$

with obvious BRST symmetry of the fields. Choosing gauge fixing function to be $\varphi = \phi$ we get

$$L_{BRST} = \bar{\psi}\mathcal{D}\psi + \bar{c}\mathcal{D}c$$

which coincides exactly with (17). The alternative choice $\varphi = \sqrt{\mathcal{D}}\psi$ leads to $L_{BRST} = \int \mathrm{d}^3x \; \bar{\phi}\phi - \bar{c}\mathcal{D}c$. The field ϕ is therefore nondynamical and we are left with

$$L_{BRST} = -\bar{c}\mathcal{D}c \tag{19}$$

The sequence of equivalences between (14), (17) and (19) shows that the dynamics of Fermi field with Lagrangian (14) is equivalent to the dynamics of Bose field with Lagrangian (19) if there are not sources for this field. Applying this result to the terms in (13) with minus sign we prove the equivalence between higher derivative and gauge invariant PV regularizations for a spinor field in the case when $m_i \neq m_j \quad \forall \; i \neq j$.

Acknowledgements

We thank BNSF contract $\Phi - 401 - 94$ for the yearly support of US dollars 50.

References

[1] L.D. Faddeev and A.A. Slavnov, *Gauge fields. Introduction to quantum theory,* 2nd edition, Benjamin 1989.

[2] S.W. Hawking, *Who's Afraid of (Higher Derivative) Ghosts?*, In Quantum Field Theory and Quantum Statistics, vol.2, p.129, ed. I.A. Batalin et al Cambridge, 1985.

[3] A. Salam and J. Stathdee, *Phys. Rev.* **D18** (1978) 4480.

[4] see, e.g., A. Bartoli and J. Julve, *Nucl. Phys.* **B425** (1994) 277 and references threin.

[5] A. Hosoya and K. Kikkawa, *Nucl. Phys.* **B101** (1975) 271 ;
J. Alfaro and P.H.Damgaard, *Ann. Phys.* (N.Y.) **202** (1990) 398.

[6] T. Kugo and S. Uehara, *Nucl. Phys.* **B197** (1982) 378 ;
L. Baulieu, *Phys. Rep.* **129** (1985) 1.